Problem 2.30

Check that Eq. 2.29 satisfies Poisson's equation, by applying the Laplacian and using Eq. 1.102.

Solution

Eq. 2.29 is the basic formula for the electric potential at $\mathbf{r} = \langle x, y, z \rangle$ due to a continuous volume charge distribution.

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\mathbf{r}')}{\mathscr{r}} \, d\tau' \tag{2.29}$$

Take the Laplacian of both sides.

$$\begin{aligned}\nabla^2 V(\mathbf{r}) &= \nabla^2 \left[\frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\mathbf{r}')}{\mathscr{r}} \, d\tau' \right] \\ &= \frac{1}{4\pi\epsilon_0} \nabla^2 \iiint \frac{\rho(\mathbf{r}')}{\mathscr{r}} \, d\tau'\end{aligned}$$

Because the integral is taken over the primed variables, ∇^2 can be brought inside the integrand. ($\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is in terms of the unprimed variables.)

$$\nabla^2 V = \frac{1}{4\pi\epsilon_0} \iiint \nabla^2 \left[\frac{\rho(\mathbf{r}')}{\mathscr{r}} \right] d\tau'$$

ρ is a function of the primed variables, so it can be pulled in front.

$$\nabla^2 V = \frac{1}{4\pi\epsilon_0} \iiint \rho(\mathbf{r}') \nabla^2 \left(\frac{1}{\mathscr{r}} \right) d\tau'$$

Use Eq. 1.102, which gives the Laplacian of $1/\mathscr{r}$.

$$\begin{aligned}\nabla^2 V &= \frac{1}{4\pi\epsilon_0} \iiint \rho(\mathbf{r}') \left[-4\pi\delta^3(\boldsymbol{\mathscr{r}}) \right] d\tau' \\ &= -\frac{1}{\epsilon_0} \iiint \rho(\mathbf{r}')\delta^3(\boldsymbol{\mathscr{r}}) \, d\tau' \\ &= -\frac{1}{\epsilon_0} \iiint \rho(\mathbf{r}')\delta^3(\mathbf{r} - \mathbf{r}') \, d\tau' \\ &= -\frac{1}{\epsilon_0} \iiint \rho(\mathbf{r}') \left[\frac{1}{|-1|^3} \delta^3(\mathbf{r}' - \mathbf{r}) \right] d\tau' \\ &= -\frac{1}{\epsilon_0} \iiint \rho(\mathbf{r}')\delta^3(\mathbf{r}' - \mathbf{r}) \, d\tau' \\ &= -\frac{1}{\epsilon_0} \rho(\mathbf{r}) \\ &= -\frac{\rho}{\epsilon_0}\end{aligned}$$

Eq. 1.98 on page 50 was used to evaluate the integral, and Eq. 1.94 on page 48 was used to alter the delta function argument.